

Quiver Grassmannians for Wild Acyclic Quivers

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Abstract. A famous result of Zimmermann-Huisgen, Hille and Reineke asserts that any projective variety occurs as a quiver Grassmannian for a suitable representation of some wild acyclic quiver. We show that this happens for *any* wild acyclic quiver.

Let k be an algebraically closed field, and Q a finite acyclic quiver. The modules which we consider are the (finite-dimensional) kQ -modules, where kQ is the path algebra of Q , thus the (finite-dimensional) representations of Q (with coefficients in k). We denote by $\text{mod } kQ$ the corresponding module category.

Let M be a representation of Q and \mathbf{d} a dimension vector for Q . The *quiver Grassmannian* $\mathbb{G}_{\mathbf{d}}(M)$ is the set of submodules of M with dimension vector \mathbf{d} ; this is a projective variety. A famous result of Zimmermann-Huisgen, Hille and Reineke asserts that any projective variety occurs as the quiver Grassmannian for some wild acyclic quiver Q , see for example [R2]. In [R3] we have shown that one may take as quiver a Kronecker quiver $Q = K(n)$, for N a suitable reduced representation of Q and as \mathbf{d} the dimension vector $(1, 1)$, see also [H]. Here, a representation of a Kronecker quiver is called *reduced* in case it has no simple injective direct summand. For a reduced representation N of a Kronecker quiver, $\mathbb{G}_{(1,1)}(N)$ is the set of indecomposable submodules of N of length 2. We call indecomposable modules of length 2 *bristles*, and $\beta(N) = \mathbb{G}_{(1,1)}(N)$ the *bristle variety* of N . We use the result of [R3] in order to show:

Theorem. *If Q be a wild acyclic quiver, then any projective variety occurs as a quiver Grassmannian for a suitable representation of Q .*

For the proof, we will construct full exact subcategories \mathcal{E} of $\text{mod } kQ$ which are equivalent to $\text{mod } kK(n)$ with n arbitrarily large. In order to define such an \mathcal{E} , we start with a pair X, Y of orthogonal bricks with $\dim_k \text{Ext}^1(Y, X) = n$, and $\mathcal{E} = \mathcal{E}(Y, X)$ will be the full subcategory of all kQ -modules M with an exact sequence of the form

$$0 \rightarrow X^a \rightarrow M \rightarrow Y^b \rightarrow 0,$$

where a, b are natural numbers. Always, \mathbf{x} and \mathbf{y} will denote the dimension vectors of X and Y , respectively. An equivalence between $\text{mod } kK(n)$ and \mathcal{E} is given by an exact fully faithful functor

$$\eta: \text{mod } kK(n) \rightarrow \text{mod } kQ$$

with image \mathcal{E} . We say that a module M in \mathcal{E} is \mathcal{E} -*reduced* provided it has no direct summand isomorphic to Y , thus provided it is the image of a reduced $kK(n)$ -module under η . The

indecomposable kQ -modules U with an exact sequence of the form $0 \rightarrow X \rightarrow U \rightarrow Y \rightarrow 0$ will be called \mathcal{E} -bristles (of course, they are the images under η of the bristles in $\text{mod } kK(n)$, note that \mathcal{E} -bristles have dimension vector $\mathbf{x} + \mathbf{y}$).

For any $kK(n)$ -module N , the functor η identifies the bristle variety $\beta(N)$ of N with the set of submodules of ηN which are \mathcal{E} -bristles. It remains to specify conditions such that the set of \mathcal{E} -bristles is just the quiver Grassmanian $\mathbb{G}_{\mathbf{x}+\mathbf{y}}(\eta N)$. We will choose X, Y so that the following closure condition (C) is satisfied:

(C) *If M is an \mathcal{E} -reduced module in $\mathcal{E}(Y, X)$ and U is a submodule of M with $\mathbf{dim} U = \mathbf{dim} X + \mathbf{dim} Y$, then U is an \mathcal{E} -bristle.*

If the condition (C) is satisfied, then for any reduced representation N of $K(n)$, there is a canonical bijection between $\mathbb{G}_{(1,1)}(N)$ and $\mathbb{G}_{\mathbf{x}+\mathbf{y}}(\eta N)$. Namely, if B is a submodule of the $kK(n)$ -module N with $\mathbf{dim} B = (1, 1)$, then ηB is a submodule of ηN with dimension vector $\mathbf{x} + \mathbf{y}$. Conversely, if U is a submodule of ηN with $\mathbf{dim} U = \mathbf{x} + \mathbf{y}$, then, by condition (C), U belongs to $\mathcal{E}(Y, X)$, say $U = \eta B$ for some $K(n)$ -submodule B and the dimension vector of B is $(1, 1)$.

Our aim is to exhibit for any wild acyclic quiver Q and any natural number m an orthogonal pair X, Y of kQ -modules which are bricks such that $\dim_k \text{Ext}^1(Y, X) = n \geq m$ and such that the condition (C) is satisfied. The following well-known proposition suggests to deal with two different cases.

Proposition. *A wild acyclic quiver Q with at least 3 vertices has a vertex ω which is a sink or a source such that the quiver Q' obtained from Q by deleting ω is connected and representation-infinite.* \square

Case 1. Assume that Q is a connected quiver with a vertex ω which is a sink or a source such that the quiver Q' obtained from Q by deleting ω is connected and representation-infinite. Up to duality, we can assume that ω is a source, thus there is an arrow $\omega \rightarrow p$ with $p \in Q'_0$.

Let $Y = S(\omega)$. Since Q' is connected and representation-infinite, there is an exceptional kQ' -module X with $\dim_k X_p \geq m$. The arrow $\omega \rightarrow p$ shows that $\dim_k \text{Ext}^1(Y, X) \geq \dim_k X_p$. This pair X, Y is the orthogonal pair of bricks which we use in order to look at $\mathcal{E}(Y, X)$.

Lemma 1. *Let a be a natural number. Any submodule W of X^a with $\mathbf{dim} W = \mathbf{x}$ is isomorphic to X .*

Proof. We denote by $\langle -, - \rangle$ the bilinear form on the Grothendieck group $K_0(kQ)$ with $\langle \mathbf{dim} M, \mathbf{dim} M' \rangle = \dim_k \text{Hom}(M, M') - \dim_k \text{Ext}^1(M, M')$. Since X is exceptional, we have $\langle X, W \rangle = \langle X, X \rangle > 0$. Therefore, there is a non-zero homomorphism $f: X \rightarrow W$. Let $\iota: W \rightarrow X^a$ be the inclusion map. The composition $\iota f: X \rightarrow X^a$ is nonzero. Since X is a brick, we see that $f: X \rightarrow W$ is a split monomorphism, in particular injective. Now $\mathbf{dim} X = \mathbf{dim} W$ implies that f is an isomorphism. \square

Proof of condition (C). Let M be an \mathcal{E} -reduced kQ -module in $\mathcal{E}(Y, X)$, say with an exact sequence

$$0 \rightarrow X^a \xrightarrow{\mu} M \xrightarrow{\pi} Y^b \rightarrow 0.$$

Let U be a submodule of M with dimension vector $\mathbf{x} + \mathbf{y}$ and inclusion map $\iota: U \rightarrow M$. The composition $\pi\iota$ is non-zero, since otherwise U would be a submodule of X^a , but $\dim_k U_\omega = 1$ whereas $X_\omega = 0$. It follows that the image of $\pi\iota$ is isomorphic to Y . If we denote the kernel of $\pi\iota$ by W , we obtain the following commutative diagram with exact rows and vertical monomorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \longrightarrow & U & \longrightarrow & Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \iota & & \downarrow & & \\ 0 & \longrightarrow & X^a & \xrightarrow{\mu} & M & \xrightarrow{\pi} & Y^b & \longrightarrow & 0. \end{array}$$

Of course, $\mathbf{dim} W = \mathbf{x}$, thus Lemma 1 shows that W is isomorphic to X . In particular, U belongs to \mathcal{E} .

It remains to show that U is indecomposable. Otherwise, U would be isomorphic to $W \oplus U$. Thus M would have a submodule isomorphic to Y . But Y is relative injective inside \mathcal{E} , thus M would have a direct summand isomorphic to Y , in contrast to our assumption that M is \mathcal{E} -reduced. This shows that U is indecomposable, thus an \mathcal{E} -bristle. \square

Case 2. Here we consider the 3-Kronecker quiver $Q = K(3)$, with two vertices 1 and 2 and three arrows $\alpha, \beta, \gamma: 1 \rightarrow 2$. Let $\lambda_1, \dots, \lambda_n$ be pairwise different non-zero elements of k with $n \geq 2$. Let $X = X(\lambda_1, \dots, \lambda_n) = (k^n, k^n; \alpha, \beta, \gamma)$ be defined by

$$\alpha(e(i)) = e(i), \quad \beta(e(i)) = \lambda_i e(i), \quad \gamma(e(i)) = e(i+1),$$

for $1 \leq i \leq n$, where $e(1), \dots, e(n)$ is the canonical basis of k^n and $e(n+1) = e(1)$. Let $Y = (k, k; 1, 0, 0)$. We denote by Q' the subquiver of Q with arrows α, β , this is the 2-Kronecker quiver $K(2)$. For the structure of the module category of the 2-Kronecker quiver $K(2)$, see for example [R1]. The restriction of X, Y to Q' shows that $\text{Hom}(X, Y) = \text{Hom}(Y, X) = 0$. The endomorphism ring of $X|_{Q'}$ is $k \times \dots \times k$; and the only endomorphisms of $X|_{Q'}$ which commute with γ are the scalar multiplications. This shows that X is a brick. Also, it is easy to see that $\dim_k \text{Ext}^1(Y, X) = n$.

Lemma 2. *Let a be a natural number. Any submodule W of X^a with $\mathbf{dim} W$ of the form (w, w) is isomorphic to X^s for some s .*

Proof: Let $M = X^a$ and decompose $M|_{Q'} = \bigoplus_{i=1}^n M(i)$, where $\beta(x) = \lambda_i x$ for $x \in M(i)_1$. Here, we use α in order to identify M_1 and M_2 . Now we consider the submodule W of M . Note that $W|_{Q'}$ has to be regular, since it cannot have any non-zero preinjective direct summand. As a regular submodule of a semisimple regular Kronecker module it has to be a direct summand of $M|_{Q'}$, thus we have a similar direct decomposition $W = \bigoplus W(i)$, where $W(i) = W \cap M(i)$.

The linear map γ restricted to $W(i)_1$ is a monomorphism $W(i)_1 \rightarrow W(i+1)_2 = W(i+1)_1$ for $1 \leq i \leq n$; we obtain in this way a monomorphism $W(1)_1 \rightarrow W(1)_2 = W(1)_1$. This shows that all the monomorphisms $W(i)_1 \rightarrow W(i+1)_2 = W(i+1)_1$ are actually bijections. Let $\dim_k W(1)_1 = s$. It follows that W is isomorphic to X^s . \square

Proof of condition (C). Let M be an \mathcal{E} -reduced kQ -module in \mathcal{E} and let U be a submodule of M with dimension vector $\mathbf{x} + \mathbf{y} = (n+1, n+1)$ and with inclusion map $\iota: U \rightarrow M$.

Starting with the exact sequence $0 \rightarrow X^a \xrightarrow{\mu} M \xrightarrow{\pi} Y^b \rightarrow 0$ and the inclusion map $\iota: U \rightarrow M$, let W be the kernel and \overline{U} the image of $\pi\iota: U \rightarrow Y^b$. We obtain the following commutative diagram with exact rows and injective vertical maps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \longrightarrow & U & \longrightarrow & \overline{U} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \iota & & \downarrow & & \\ 0 & \longrightarrow & X^a & \xrightarrow{\mu} & M & \xrightarrow{\pi} & Y^b & \longrightarrow & 0; \end{array}$$

Let us consider the restriction of these modules to Q' . Since $M|Q'$ is regular, it has no non-zero preinjective direct summand. Thus any submodule of $M|Q'$ with dimension vector $(n+1, n+1)$ has to be regular. This shows that $U|Q'$ is regular. Actually, $M|Q'$ is semisimple regular, thus also its regular submodule $U|Q'$ is semisimple regular (and a direct summand of $M|Q'$). Next, $\pi\iota$ is a map between regular kQ' -modules, it follows that the kernel $W|Q'$ and the image $\overline{U}|Q'$ are regular kQ' -modules. In particular, the dimension vector of W is of the form $\mathbf{dim} W = (w, w)$ for some $0 \leq w \leq n+1$.

Now $U|Q'$ is a regular submodule of the semisimple regular kQ' -module $Y^b|Q'$, thus $\overline{U}|Q'$ is a direct sum of copies of $Y|Q'$. By construction, Y is annihilated by γ . Since \overline{U} is a submodule of Y^b , it follows that \overline{U} is annihilated by γ . Altogether, we see that \overline{U} is the direct sum of copies of Y .

We claim that $W \neq 0$. Otherwise $U = \overline{U} = Y^{n+1}$, thus Y is a submodule of M . But Y is relative injective in \mathcal{E} , thus Y would be a direct summand of M . However, by assumption, M is \mathcal{E} -reduced. This contradiction shows that $W \neq 0$.

Now W is a submodule of X^a with dimension vector (w, w) , thus, according to Lemma 2, W is a direct summand of say s copies of X and $s \geq 1$. The equality $(w, w) = (sn, sn)$ implies that $s = 1$, since $w \leq n+1$ and $n \geq 2$. In this way, we see that W is isomorphic to X . It follows that $\mathbf{dim} \overline{U} = (1, 1)$ and therefore $\overline{U} = Y$.

Finally, as in Case 1, we see that U is indecomposable, using again the assumption that M is \mathcal{E} -reduced. This shows that U is an \mathcal{E} -bristle. \square

Remark. We should stress that given orthogonal bricks X, Y in $\text{mod } kQ$, the condition (C) is usually not satisfied. Here is a typical example for $Q = K(3)$. As above, let $Y = (k, k; 1, 0, 0)$, but for X we now take $X = X'(\lambda_1, \lambda_2) = (k^2, k^2; \alpha, \beta, \gamma)$, defined by

$$\alpha(e(i)) = e(i), \quad \beta(e(i)) = \lambda_i e(i), \quad \gamma(e(1)) = e(2), \quad \gamma(e(2)) = 0$$

for $1 \leq i \leq 2$. Again, $e(1), e(2)$ is the canonical basis of k^2 and $\lambda_1 \neq \lambda_2$ are assumed to be non-zero elements of k . Since $\dim_k \text{Ext}^1(Y, X) = 2$, there is an equivalence $\eta: \text{mod } kK(2) \rightarrow \mathcal{E}(Y, X)$. Let N be an indecomposable $kK(2)$ -module with dimension vector $(2, b)$ (note that b has to be equal to 1, 2 or 3) and $M = \eta N$. Thus there is an exact sequence

$$0 \rightarrow X^2 \rightarrow M \rightarrow Y^b \rightarrow 0.$$

Since we assume that N is indecomposable, it is reduced, thus M is \mathcal{E} -reduced. Note that X has a (unique) kQ -submodule V with dimension vector $(1, 1)$: the vector spaces V_1 and V_2 both are generated by $e(2)$. The submodule $U = X \oplus V$ of X^2 is a submodule of M

with dimension vector $(3, 3) = \mathbf{x} + \mathbf{y}$, and it is not an \mathcal{E} -bristle. Thus, condition (C) is not satisfied. Here, η defines a proper embedding of $\beta(N) = \mathbb{G}_{(1,1)}(N)$ into $\mathbb{G}_{\mathbf{x}+\mathbf{y}}(M)$.

References

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